

Some results on general quadratic reflected BSDEs

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Abstract

We study reflected BSDE in a general continuous martingale setting, when the coefficient f of the driver has at most quadratic growth in the control variable Z , with a bounded terminal condition and a lower obstacle which is bounded above. We obtain the basic results in this setting : comparison and uniqueness, existence, stability. For the comparison theorem, and the special comparison theorem for reflected BSDEs (which allows one to compare the increasing processes of two solutions), we give straightforward, intrinsic proofs which do not rely on comparison for standard BSDEs. We obtain existence by using the fixed point theorem and then a series of perturbations. While we first show existence under the assumption that f is Lipschitz in the primary variable Y , we then extend this to the case where f can have slightly-superlinear growth and the case where f is monotonous in Y with arbitrary growth. We also obtain a local Lipschitz estimate in BMO for the martingale part of the solution.

1 Introduction

Since Pardoux and Peng initiated the systematic study of (non-linear) backward stochastic differential equations (BSDEs thereafter) in [23], these equations have been proved useful for a number of areas. They are intimately linked with the stochastic version of the maximum principle for stochastic optimal control problems, they provide probabilistic representations for the solution to some partial differential equations (PDEs thereafter), and they are natural in finance where, beyond coming in through the use of the above fields, they provide the natural language in which the replication of a European option is expressed. Reflected BSDEs (RBSDEs) have applications in the same areas, where they are linked to optimal stopping, PDEs with obstacle (variational inequalities) and American options.

“Reflected” BSDEs are BSDEs subject to a constraint : the solution process Y is required to remain above a lower obstacle L . In order to achieve this, it is necessary to add to the usual dynamics $dY_s = -f_s ds + Z_s dW_s$ a “force” dK that drives Y upward. One wants that extra term to be minimal, so that K is only active to prevent Y from passing below the obstacle L . This optimality condition (the Skorohod condition) is therefore expressed as $\int_0^T 1_{\{Y_s > L_s\}} dK_s = 0$. So in the end a reflected BSDE takes the following form :

$$\left\{ \begin{array}{l} dY_s = -f(s, Y_s, Z_s) ds - dK_s + Z_s dW_s, \\ Y_T = \xi, \\ Y_t \geq L_t \text{ for all } t \in [0, T], \\ K \text{ is continuous, increasing, starts from 0 and } \int_0^T 1_{\{Y_s > L_s\}} dK_s = 0, \end{array} \right. \quad (1)$$

where the solution to be determined is now the triple (Y, Z, K) . In this paper, f will be assumed to have at most quadratic growth in z . The terminal condition ξ is assumed to be bounded and the obstacle process L a continuous semimartingale bounded above.

Reflected BSDEs were introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [13]. These authors considered the case where f is Lipschitz, the terminal condition is square-integrable and the lower obstacle a continuous square-integrable semimartingale, the natural extension of Pardoux and Peng [23]. Given the similarity between reflected BSDEs and standard BSDEs, most developments made in the theory of BSDEs can be adapted to reflected BSDEs, in particular on the problem of well-posedness. An account of this can be found in Bayraktar and Yao [3]. The first existence, uniqueness and comparison results for quadratic BSDEs (that is, when the coefficient f is allowed to have a quadratic growth in z) were obtained by Kobylanski in [17], under the assumption that the terminal condition ξ is bounded. These results have since then been extended to more general cases, for instance by allowing f to have slightly superlinear growth in y in Lepeltier and San Martin [19], or ξ to be exponentially integrable in Briand and Hu [7]. Those results have been obtained for reflected BSDEs as well, see Kobylanski, Lepeltier, Quenez and Torrès [18] and Lepeltier and Xu [21]. Work has also been done regarding the assumptions on the obstacle L , for instance Peng and Xu [25] worked with L^2 obstacles, and in the context of quadratic f and exponentially integrable ξ Bayraktar and Yao [3] removed the condition that L be bounded. However, the case of quadratic BSDEs remains significantly more difficult than that of Lipschitz BSDEs, and the methods used initially are often quite involved. Recently, Tevzadze gave in [26] a new and simpler treatment for the existence of solutions to a quadratic BSDEs, and again, more recently, Briand and Elie gave a new approach in [6].

Backward stochastic problems in general can be considered outside a Brownian setting. For instance, BSDEs have been studied in a general martingale setting in El Karoui and Huang [12], and in a general filtered probability space in Cohen and Elliott [9]. While papers dealing with the theory for reflected BSDE are set in a Brownian setting, we consider here reflected BSDEs in a general continuous-martingale setting and obtain the basic results (comparison and uniqueness, existence, stability) using simple methods, intrinsic to RBSDEs.

We first show, in section 3, how comparison for quadratic reflected BSDEs can be proved directly, as in El Karoui *et al.* [13], using the BMO argument from Hu, Imkeller and Müller [11], as opposed to via an optimal stopping representation and comparison for BSDEs (see [18]). This result holds naturally for f locally Lipschitz in y , instead of globally Lipschitz. We then give a new proof of the special comparison theorem for reflected BSDEs, which allows one to compare the increasing processes when one RBSDE solution dominates another. This theorem was first proved in Hamadène, Lepeltier and Matoussi [10], and reused in Peng and Xu [25]. In the papers ([10], [25], [20], [18]) where it appears, the proof relies on the penalization approach and the comparison theorem for standard BSDEs, comparing quantities which, at the limit, become the increasing processes. The proof we give here is more related to the non-linear Snell envelope view of reflected BSDEs, and holds without the regularity assumptions usually made on f .

In section 4 we show how the perturbative technique introduced by Tevzadze [26] can be adapted to reflected BSDEs to prove existence of a solution. When trying to solve a quadratic BSDE with the fixed point theorem, one can conclude only when the data $(f(\cdot, 0, 0), \xi)$ are sufficiently small ($f(s, 0, 0)ds$ is the residual drift, that drives the solution even if $(Y_s, Z_s) = 0$). One can then build on this result to construct a solution in the general case.

This technique can be understood as a type of “vertical” splitting and recombination, and is in that sense an analogue to what is done for Lipschitz BSDEs. In that classical setting, if one works with the natural norm on the space where one looks for solutions, which in that context is the space of square-integrable processes, one finds that the fixed point theorem applies if the time interval is small enough. A natural way to use this is then to split a general time interval into pieces small enough that one can obtain a solution on each interval, and patch them together to obtain a solution on the whole interval. For quadratic

BSDEs, since one can apparently solve the BSDE only for small data, the idea is to split a general set of data into pieces small enough that one can obtain a solution for each piece, and then combine them to obtain a solution to the initial problem (see also [27]). One can also understand this method as a series of perturbations. One first solves a BSDE with small data, then successively solves perturbation equations and adds the associated solutions, allowing the size of the data to grow at each step. At the end, one has built a solution to the initial BSDE with macroscopic data.

In order to obtain a solution to a backward stochastic equation (with small data) via the fixed point theorem, one mainly needs to understand the underlying backward stochastic problem (for BSDEs, this underlying problem is the semimartingale decomposition, for reflected BSDEs, it is a Snell envelope problem). However, for the perturbation procedure to work well, the underlying problem should be a linear problem (for instance, it has been successfully applied recently in Kazi-Tani, Possamai and Zhou [16] to quadratic BSDEs with jumps). This way, the equations satisfied by the perturbations are of the same nature as the equations satisfied by the solutions. This is not the case for reflected BSDEs, as the underlying problem is an optimization problem, and therefore non-linear. It is however possible to identify the equation that a perturbation should satisfy. The obstacle cannot be perturbed easily during the procedure, but this can be dealt with by assuming from the start that it is negative, a case which covers all the others by a simple translation. In particular, unlike in [18], we don't need L to be bounded but only require it to be upper bounded.

We then study the stability of the solution with respect to changes in the terminal condition ξ and in the residual drift $f(\cdot, 0, 0)$. This stability can be obtained as a direct consequence of the use of the fixed point theorem for showing existence (uniqueness being acquired at that stage). We further obtain for the martingale part of the solution a local Lipschitz bound in the space BMO . Global Lipschitz bounds in \mathcal{H}^p were obtained already in Briand, Delyon, Hu, Pardoux and Stoica [5] (see also Briand and Confortola [4], Ankirchner, Imkeller and Dos Reis [1]). Kazi-Tani, Possamai and Zhou [16] provide a global $\frac{1}{2}$ -Hölder estimate in the smaller space BMO . Here, we can obtain a stronger regularity for small perturbations.

In section 5, we extend the scope of the existence theorem of section 4. In the usual setting where f is a Lipschitz function of y , the sequence of perturbations described above can be performed uniformly without problem. When the first derivative f_y is not a bounded function of y , the maximal allowed size for a perturbation depends on the size of the solution to the reflected BSDE that one wants to perturb, so it is not clear *a priori* that the procedure would terminate after finitely many perturbations. We show, however, that this is the case as soon as one can obtain an *a priori* bound for Y in S^∞ , thus recovering the generality on f considered in Kobylanski *et al.* [18] —slightly superlinear growth— and Xu [28] —monotonicity and arbitrary growth. This monotonicity assumption, introduced by Pardoux (see for instance [22]) in the framework of BSDEs with Lipschitz-dependence in z , and later generalized to coefficients f quadratic in z in Briand, Lepeltier and San Martin [8], is relevant for reaction-diffusion equations.

In the following section, we motivate the general continuous-martingale setting, and specify the notation and the framework that will be used throughout the paper.

2 Setting

General, continuous-martingale setting.

The reflected BSDE (1) is set in a Brownian setting. Even in this setting, it is sometimes useful to consider that the reference increasing process is not the time ds . For instance, Pardoux and Zhang observed in [24] that when looking at the BSDE associated with a semilinear parabolic PDE in a domain with Neumann boundary conditions (so-called “generalized BSDEs”), the drift term is of the form $f(Y_s, Z_s)ds + f'(Y_s)dA_s$ where A is the local time of the underlying

diffusion on the regular boundary, and is therefore orthogonal to the Lebesgue measure. In that case, one can enhance the increasing process by setting $dC = ds + dA$, and find f'' such that $f(Y_s, Z_s)ds + f'(Y_s)dA_s = f''(Y_s, Z_s)dC$ (see El Karoui and Huang [12]).

The reference martingale M doesn't have to be a Brownian motion, and in particular doesn't have to enjoy the martingale representation property (see El Karoui, Peng and Quenez [14]). The martingale part N of the solution then has the decomposition $N = \int Z dM + N^\perp$ on the reference martingale M , with N^\perp orthogonal to M (i.e. $\langle N^\perp, M \rangle = 0$). The quadratic variation $d\langle M \rangle$ of M is assumed to be absolutely continuous (component-wise) with respect to dC (if it is not, one can enhance dC so that it becomes the case). Write then $d\langle M \rangle = a dC = \sigma \sigma^* dC$. Since Z_s is uniquely determined only when no component of $d\langle M \rangle_s$ is zero, the drift term will only depend on $Z\sigma$. The orthogonal component N^\perp of the martingale part N is meaningful in the context of Föllmer-Schweizer strategies in incomplete markets (see [14]). One wants in that context to have in the drift a quadratic term $g_s d\langle N^\perp \rangle_s$. By the perturbative nature of the method used later to prove existence, one is naturally led to consider also a term $d\langle \nu, N^\perp \rangle$ linear in N^\perp .

So in the end, we will study the following general reflected BSDE :

$$\begin{cases} dY_s = -dV(Y, N)_s - dK_s + dN_s, \\ Y_T = \xi, \\ Y_t \geq L_t \text{ for } t \leq T, \text{ and} \\ K \text{ increasing, continuous, starting from 0 and such that } 1_{\{Y_s > L_s\}} dK_s = 0 \end{cases} \quad (2)$$

where the drift is given by

$$dV(Y, N)_s = f(s, Y_s, Z_s \sigma_s) dC_s + d\langle \nu, N^\perp \rangle_s + g_s d\langle N^\perp \rangle_s.$$

This is referred to as the reflected BSDE of parameters/data $(V, \xi, L) = (f, \nu, g, \xi, L)$.

The framework is a filtered probability space $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t \in [0, +\infty[}, P)$ satisfying the usual conditions. T is an \mathcal{F} -stopping time valued in \mathbb{R}_+ (finite stopping time). The continuous square-integrable martingale M is assumed to be *BMO* (see below). All the processes considered are continuous.

C is a continuous and progressively measurable increasing process (starting from 0) such that, roughly, all the finite variational processes which are related to parameters (not depending on the solution) are absolutely continuous with respect to it. In particular, $d\langle M \rangle_s = a_s dC_s = \sigma_s \sigma_s^* dC_s$. It is assumed that the positive symmetric matrix a (or equivalently σ) is bounded away from 0 and infinity (i.e. bounded and uniformly elliptic).

The parameters of the BSDE (coefficients f, ν, g of the drift V , terminal condition ξ , obstacle L) are as follows :

- $f : \Omega \times [0, T] \times \mathbb{R} \times M_{1,d}(\mathbb{R}) \rightarrow \mathbb{R}$ is $Prog \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(M_{1,d}(\mathbb{R}))$ -measurable, where $M_{n,d}(\mathbb{R})$ is the space of $n \times d$ matrices with entries in \mathbb{R} . $Prog = Prog(\mathcal{F}_T)$ is the progressively measurable sigma-field on the interval $[0, T]$ (the set of pairs (ω, t) such that $t \leq T(\omega)$).
- ν is a BMO martingale orthogonal to M (that is $\langle \nu, M \rangle = 0$)
- g is a progressively measurable and bounded scalar process.
- ξ is an \mathcal{F}_T -measurable, bounded random variable.
- L is a continuous semimartingale bounded above.

Throughout the paper, we assume that f has at most quadratic growth in the variable z , in the following sense :

- (**A_{qg}**) There exists a growth function $\lambda(\cdot)$ (i.e. $\lambda : \mathbb{R} \rightarrow \mathbb{R}_+$ symmetric, increasing on \mathbb{R}_+ , bounded below by 1) and a positive process $h \in L^2_{BMO}$ (i.e. $\int h dM \in BMO$, see below) such that :

$$|f(t, y, z)| \leq \lambda(y)(h_t^2 + |z|^2).$$

The assumption as written above allows for any growth in y , although most of the time f will have at most linear growth in y .

Solutions to the reflected BSDE.

A solution to the reflected BSDE is generally understood as a triple $S = (Y, N, K)$ where Y is a semimartingale, N a square-integrable martingale ($\in \mathcal{H}^2$) and K an increasing process (starting from 0), such that (2) is satisfied, with $N = \int Z dM + N^\perp$.

Note that a solution can also be understood as a pair $S = (Y, N)$ such that, defining K from $K_0 = 0$ and the dynamics equation in (2), K is indeed found to be increasing and satisfies the Skorohod condition. This will often be what is meant by solution in the rest of the paper.

Under the assumption of quadratic growth and bounded terminal condition, we consider only bounded solutions : $Y \in \mathcal{S}^\infty$. For those, N is found to be a BMO martingale (see the a priori estimate of proposition 1 below). So a solution will always be understood as being in $\mathcal{S}^\infty \times BMO(\times \mathcal{A})$.

Spaces of processes.

We will make use of the following spaces.

- $BMO(P)$ is the space of all the BMO P -martingales, that is those for which the norm

$$\|N\|_{BMO(P)}^2 = \sup_{t \in \mathcal{T}_0^T} \|E_P(\langle N \rangle_T - \langle N \rangle_t | \mathcal{F}_t)\|_\infty$$

is finite, where \mathcal{T}_0^T is the set of stopping times t such that $0 \leq t \leq T$. The mention of the measure P will be omitted whenever no confusion is possible. When $X \in BMO$, then $\mathcal{E}(X)$ is a UI martingale, so one can define a measure Q by stating that on \mathcal{F}_t , $\frac{dQ}{dP} = \mathcal{E}(X)_t$. Also, we will use frequently the fact that for any $N \in BMO(P)$, $\tilde{N} = N - \langle X, N \rangle$ is in $BMO(Q)$ (cf Kazamaki [15], theorems 2.3 and 3.3).

- L^2_{BMO} is the space of processes h such that $\int h dM \in BMO$.
- $L^{\infty,2}$ is the space of processes x such that $\int_0^T |x_s|^2 dC_s \in L^\infty$, and $L^{\infty,1}$ is that of processes such that $\int_0^T |x_s| dC_s \in L^\infty$, with norms $\|x\|_{\infty,2}^2 = \|\int_0^T |x_s|^2 dC_s\|_\infty$ and $\|x\|_{\infty,1} = \|\int_0^T |x_s| dC_s\|_\infty$.

The first space will be used for the process r appearing in many growth assumptions on y later on. It is there to take into account the fact that C_T might not be bounded (in facts, if C incorporates a local time of a diffusion on a regular boundary, it has exponential moments but is not bounded). In the case where $dC_s = ds$ and T is a constant (or a majorant for the bounded terminal time), one would have $r = 1$ and $\|r\|_{\infty,2}^2 = T$.

- \mathcal{A} is the space of accumulators, that is : progressively measurable, continuous, increasing processes starting from 0.

BMO property for N .

Proposition 1. *Let f satisfy (\mathbf{A}_{qg}) , $\nu \in BMO$ and g be bounded. Let Y be a continuous semimartingale, N be a square-integrable martingale and K be an increasing process such that Y has the decomposition :*

$$dY = -dV(Y, N) - dK + dN,$$

where $dV(Y, N)_s = f(s, Y_s, Z_s \sigma_s) dC_s + d\langle \nu, N^\perp \rangle_s + g(s) d\langle N^\perp \rangle_s$. If Y is bounded (i.e. $Y \in \mathcal{S}^\infty$), then $N \in BMO$ and $K \in \mathcal{A}_{BMO}$.

Here, \mathcal{A}_{BMO} refers to increasing the processes $K \in \mathcal{A}$ such that the quantity $\|K\|_{\mathcal{A}_{BMO}} = \sup_t \|E(K_T - K_t | \mathcal{F}_t)\|_\infty$ is finite. Note that this statement is, to some extent, not so much about solutions to a (possibly reflected) BSDE but about quadratic semimartingales (see Barrieu and El Karoui [2]), and quadratic semimartingales are considered here up to a monotonous process.

Remark 2. The result implies in particular the following : if Y is a bounded semimartingale with decomposition $dY = -dV - dK + dN$, with K monotonous (which boils down to increasing, up to considering $-Y$) and if the process V is in L^1_{BMO} (i.e. $\sup_t \|E(\int_t^T |dV_s| | \mathcal{F}_t)\|_\infty < +\infty$), then $N \in BMO$ and $K \in \mathcal{A}_{BMO}$.

Proof. The proof uses the usual exponential transform. Let $\mu \in \mathbb{R}$, whose sign and value will be chosen later. By Itô's formula for the process $\exp(\mu Y)$ between a stopping time $t \in \mathcal{T}_0^T$ and T one has

$$\begin{aligned} e^{\mu Y_t} - \mu \int_t^T e^{\mu Y_s} dK_s + \frac{\mu^2}{2} \int_t^T e^{\mu Y_s} d\langle N \rangle_s &= e^{\mu Y_T} + \mu \int_t^T e^{\mu Y_s} dV_s \\ &\quad - \mu \int_t^T e^{\mu Y_s} dN_s. \end{aligned} \quad (3)$$

Since $Y \in \mathcal{S}^\infty$, the process $e^{\mu Y}$ is bounded, and since N is a square-integrable martingale, $\int e^{\mu Y} dN$ is a martingale. We have

$$|dV_s| \leq |f(s, Y_s, Z_s \sigma_s)| dC_s + |d\langle \nu, N^\perp \rangle_s| + |g(s)| d\langle N^\perp \rangle_s,$$

and using the quadratic growth assumption on f we have

$$|f(s, Y_s, Z_s \sigma_s)| \leq \lambda(Y_s)(h_s^2 + |Z_s \sigma_s|^2) \leq \Lambda(h_s^2 + |Z_s \sigma_s|^2),$$

where $\Lambda = \lambda(\|Y\|_{\mathcal{S}^\infty})$. In order to make the following computations lighter, we use the notation $|\cdot|_{[t]}^2$ for the weighted squared quantity : $E(\int_t^T e^{\mu Y_s} \dots | \mathcal{F}_t)$, where \dots is replaced by the appropriate quantity depending on whether one is measuring h , Z , or N^\perp and N . Using the Kunita-Watanabe inequality and $ab \leq a^2 + b^2$, we see that

$$E\left(\int_t^T e^{\mu Y_s} |dV_s| | \mathcal{F}_t\right) \leq \Lambda \left[|h|_{[t]}^2 + |Z|_{[t]}^2 \right] + |\nu|_{[t]}^2 + |N^\perp|_{[t]}^2 + \|g\|_\infty |N^\perp|_{[t]}^2.$$

Recall that by the orthogonality of M and N^\perp , $d\langle N \rangle = |Z\sigma|^2 dC + d\langle N^\perp \rangle$, and therefore both $|Z\sigma|^2 dC$ and $d\langle N^\perp \rangle$ are less than or equal to $d\langle N \rangle$. So, setting $b = (\Lambda + 1 + \|g\|_\infty)$, and taking the conditional expectation of (3) with respect to \mathcal{F}_t , one has

$$0 - \mu |K|_{[t]} + \left\{ \frac{\mu^2}{2} - |\mu|b \right\} |N|_{[t]}^2 \leq e^{|\mu| \|Y\|} + |\mu| \left[\Lambda |h|_{[t]}^2 + |\nu|_{[t]}^2 \right] - 0.$$

We now choose $\mu = -4b$. Since $b \geq 1$, $b^2 \geq b$. We deal with the weights by saying that $e^{-|\mu|\|Y\|} \leq e^{\mu Y_s} \leq e^{|\mu|\|Y\|}$. Taking the the \sup_t , we obtain finally

$$\|K\|_{\mathcal{A}_{BMO}} + \|N\|_{BMO}^2 \leq \frac{e^{8b\|Y\|_\infty}}{2b} \left[1 + 4b(\Lambda\|h\|_{L_{BMO}^2}^2 + \|\nu\|_{BMO}^2) \right] < +\infty.$$

□

Note however that while it indeed gives a bound for $N \in BMO$, this estimate is not fully satisfying in that it does not guarantee that if $\|Y\|_{\mathcal{S}^\infty} \rightarrow 0$ then $\|N\|_{BMO} \rightarrow 0$. One could also provide an a priori estimate for $\|Y\|_{\mathcal{S}^\infty}$, but that will not be used in the paper. More precisely, the standard assumption under which existence is proved in section 4 is enough to carry the perturbation procedure. In section 5 however, we consider more general cases and there we will need the related a priori estimates.

3 Comparison theorems and uniqueness.

3.1 Comparison theorem.

We give below the comparison theorem in our setting, which guarantees uniqueness in the existence theorem 11. The regularity assumption that we require for the theorem to hold is, for notational simplicity, the following :

(**A_{Df}**) The function f is of class \mathcal{C}^1 (in the variable (y, z) , for all ω, t) with

$$|f_y(t, y, z)| \leq \rho(y)r_t^2 \quad \text{and} \quad |f_z(t, y, z)| \leq \rho'(y)(h_t + |z|),$$

for some growth function ρ and ρ' .

Theorem 3. *Consider two sets of parameters (f, ν, g, ξ, L) and (f', ν, g', ξ', L') , and assume that :*

1. *there exist solutions (Y, N, K) and (Y', N', K') to the corresponding reflected BSDEs,*
2. *the parameters are ordered : $f' \leq f$, $g' \leq g$, $\xi' \leq \xi$ and $L' \leq L$,*
3. *f is regular enough : it satisfies (**A_{Df}**).*

Then one has $Y' \leq Y$.

Proof. The proof relies on a classical linearization argument and the properties of solutions to a linear BSDE, just like when proving comparison for BSDEs. More precisely, we study the positive part $(\Delta Y)^+$, where $\Delta X = X' - X$ for a generic quantity X , and show that $(\Delta Y)^+ \leq 0$.

Denoting by l the local time of ΔY in 0, the Itô-Tanaka formula gives

$$\begin{aligned} d(\Delta Y)_s^+ &= 1_{\{\Delta Y_s > 0\}} d\Delta Y_s + \frac{1}{2} dl_s \\ &= 1_{\{\Delta Y_s > 0\}} \left[-d\Delta V_s - d\Delta K_s + d\Delta N_s \right] + \frac{1}{2} dl_s. \end{aligned} \tag{4}$$

Now, gathering terms, rewriting differences, and linearizing some,

$$\begin{aligned}
d\Delta V_s &= \left[f'(s, Y'_s, Z'_s \sigma_s) - f(s, Y_s, Z_s \sigma_s) \right] dC_s \\
&\quad + d\langle \nu, (N')^\perp - N^\perp \rangle + g'_s d\langle (N')^\perp \rangle_s - g_s d\langle N^\perp \rangle_s \\
&= \left[(\Delta f)(s, Y'_s, Z'_s \sigma_s) + f(s, Y'_s, Z'_s \sigma_s) - f(s, Y_s, Z_s \sigma_s) \right] dC_s \\
&\quad + d\langle \nu, \Delta N^\perp \rangle_s + (\Delta g)_s d\langle (N')^\perp \rangle_s + g_s \left[d\langle (N')^\perp \rangle - d\langle N^\perp \rangle_s \right] \\
&= \left[(\Delta f)(s, Y'_s, Z'_s \sigma_s) + f(s, Y'_s, Z'_s \sigma_s) - f(s, Y_s, Z'_s \sigma_s) + f(s, Y_s, Z'_s \sigma_s) - f(s, Y_s, Z_s \sigma_s) \right] dC_s \\
&\quad + d\langle \nu, \Delta N^\perp \rangle_s + (\Delta g)_s d\langle (N')^\perp \rangle_s + g_s d\langle (N')^\perp + N^\perp, \Delta N^\perp \rangle \\
&= \left[\Delta f + F_y \Delta Y + F_z \Delta Z \sigma \right] dC \\
&\quad + d\langle \nu', \Delta N^\perp \rangle + (\Delta g) d\langle (N')^\perp \rangle,
\end{aligned}$$

where

$$\begin{aligned}
F_y(s) &= F_y(s, Y_s, Y'_s, Z'_s \sigma_s) = \int_0^1 f_y(s, Y_s + u\Delta Y_s, Z'_s \sigma_s) du, \\
F_z(s) &= F_z(s, Y_s, Z_s \sigma_s, Z'_s \sigma_s) = \int_0^1 f_z(s, Y_s, Z_s \sigma_s + u\Delta Z_s \sigma_s) du \text{ and} \\
\nu' &= \nu + \int g d(N^\perp + (N')^\perp).
\end{aligned}$$

So we can rewrite (4) as

$$d(\Delta Y)^+ = -dD - F_y(\Delta Y)^+ dC + 1_{\{\Delta Y > 0\}} \left[d\Delta N - F_z \Delta Z \sigma dC - d\langle \nu', \Delta N^\perp \rangle \right],$$

where

$$dD = 1_{\{\Delta Y > 0\}} \left[\Delta f dC_s + \Delta g d\langle (N')^\perp \rangle + d\Delta K \right] - \frac{1}{2} dl$$

is a decreasing process. Indeed $\Delta f \leq 0$, $\Delta g \leq 0$, $dl \geq 0$ and

$$1_{\{\Delta Y > 0\}} d\Delta K = \underbrace{1_{\{\Delta Y > 0\}} dK'}_{=0} - \underbrace{1_{\{\Delta Y > 0\}} dK}_{\geq 0} \leq 0$$

because $dK \geq 0$ and on $\{\Delta Y > 0\}$, $Y' > Y \geq L \geq L'$ so $dK' = 0$.

$(\Delta Y)^+$ is therefore seen as the solution to a linear equation. Define the integrating factor $B_t = e^{\int_0^t F_y(u) dC_u}$ and the measure Q by $\frac{dQ}{dP} = \mathcal{E}(\int F_z \sigma^{-1} dM + \nu')_t$. By the assumption on f_z and the fact that $h \in L^2_{BMO}$ and N, N' and ν are in BMO, $\int F_z \sigma^{-1} dM + \nu'$ is in BMO, and therefore Q is indeed well defined. Then, $\widetilde{\Delta N} = \Delta N - \int F_z \Delta Z \sigma dC_s - \langle \nu', \Delta N^\perp \rangle$ is a $BMO(Q)$ -martingale. By the assumption on f_y , the process $B \cdot 1_{\{\Delta Y > 0\}}$ is bounded so $\int B_u 1_{\Delta Y_u > 0} d\widetilde{\Delta N}_u$ is again a Q -martingale. Therefore, looking at the dynamic of $\widehat{Y} = BY$ under Q we finally find that

$$0 \leq (\Delta Y_t)^+ = E_Q \left(e^{\int_t^T F_y(v) dC_v} (\Delta \xi)^+ + \int_t^T e^{\int_t^u F_y(v) dC_v} dD_u \middle| \mathcal{F}_t \right) \leq 0.$$

□

Remark 4. The previous theorem is stated, for convenience, for a function f which is C^1 . Typically, for the comparison theorem one only requires f to be locally Lipschitz, in which case the processes F_y, F_z have to be replaced by the differential quotients : $\delta_y f(Y, Y', Z' \sigma) = \frac{f(Y', Z') - f(Y, Z')}{Y' - Y} 1_{Y \neq Y'}$, etc, and the proof works as long as $F_y \in L^{\infty,1}$ and $F_z \in L^2_{BMO}$. These criteria are satisfied as soon as

(**A**_{ILip}) There exist growth functions ρ and ρ' , and processes $r \in L^{\infty,2}$ and $h \in L^2_{BMO}$ such that

$$|f(t, y', z') - f(t, y, z)| \leq \rho(y, y') r_t^2 |\Delta y| + \rho'(y, y') (h_t + |z| + |z'|) |\Delta z| .$$

Note : when ρ, ρ' are constants this is the standard assumption of local Lipschitz regularity made in the quadratic BSDE literature (see for instance Briand and Elie [6] and Tevzadze [26]). However, since we are dealing with bounded solutions, the assumption can be weakened to the case (**A**_{ILip}) where ρ, ρ' are growth functions.

3.2 Special comparison theorem.

When the two sets of parameters are in a comparison configuration and when the lower obstacles are the same, one can say more than $Y' \leq Y$ and also compare the increasing processes of the two solutions, K' and K .

Proposition 5. *Let (f, g, ν, ξ, L) and (f', g', ν, ξ', L) be some parameters, and assume that :*

1. *there exist solutions $S = (Y, N, K)$ and $S' = (Y', N', K')$ to the corresponding RBSDEs,*
2. *the drift coefficients are ordered : $f' \leq f, g' \leq g,$*
3. *Y' is dominated by $Y : Y' \leq Y.$*

Then it is the case that $dK_t \leq dK'_t$.

The intuition is quite clear. First, since one has $Y'_t \leq Y_t$, if Y doesn't touch the barrier ($Y_t > L_t$), then $dK_t = 0$ and whether $Y'_t > L_t$ or $Y'_t = L_t$, one has $dK'_t \geq 0 = dK_t$. So the only non-trivial case is when Y touches the barrier, and therefore Y' as well. In that case, since the extra forces dK' and dK are minimal, they only prevent the drifts dV' and dV from driving the solutions Y' and Y under the obstacle. But since $dV'_t \leq dV_t$ in that case, the correction that could be needed for Y will be less than that needed for Y' . The proof makes this heuristics rigorous.

Proof. In this proof, contrary to the rest of the paper, ΔX denotes $X - X'$ for a generic quantity X . In order to deal properly with what happens locally when the process ΔY touches 0, we proceed as in El Karoui *et al.* [13] : write down the structure of ΔY and ΔY^+ , argue that these two processes are equal (since by assumption $\Delta Y \geq 0$), identify their finite variational and martingale parts, and then extract the relevant information. Our goal is to prove that $d\Delta K \leq 0$.

We have

$$\begin{aligned} d\Delta Y &= -d\Delta V - d\Delta K + d\Delta N \quad \text{and} \\ d(\Delta Y)^+ &= 1_{\{\Delta Y > 0\}} d\Delta Y + \frac{1}{2} dl , \end{aligned}$$

where l is the local time of ΔY at 0. Identifying the finite variational and martingale parts, we see that

$$\begin{aligned} -d\Delta V - d\Delta K &= 1_{\{\Delta Y > 0\}} \left(-d\Delta V - d\Delta K \right) + \frac{1}{2} dl \quad \text{and} \\ d\Delta N &= 1_{\{\Delta Y > 0\}} d\Delta N , \end{aligned}$$

that is to say

$$\begin{aligned} 1_{\{\Delta Y=0\}} \left(-d\Delta V - d\Delta K \right) &= \frac{1}{2}dl \quad \text{and} \\ 1_{\{\Delta Y=0\}} d\Delta N &= 0 . \end{aligned}$$

The second equation implies, by Itô's isometry and the orthogonality between M and ΔN^\perp , that $1_{\Delta Y=0} \left(|\Delta Z \sigma|^2 dC + d\langle \Delta N^\perp \rangle \right) = 0$. So we know than on the set $\{Y' = Y\}$ (i.e. against $1_{\{\Delta Y=0\}}$) we have $Y = Y'$ and $Z = Z'$. We also notice that by the Kunita-Watanabe inequality, $1_{\{\Delta Y=0\}} d\langle \nu', \Delta N^\perp \rangle = 0$ (for any continuous semimartingale ν' that could possibly come up).

The drift term can be rewritten

$$\begin{aligned} d\Delta V_t &= (f(S) - f'(S'))dC + d\langle \nu, N^\perp \rangle - d\langle \nu, (N')^\perp \rangle + gd\langle N^\perp \rangle - g'd\langle (N')^\perp \rangle \\ &= (f(S) - f(S') + (\Delta f)(S'))dC + d\langle \nu, \Delta N^\perp \rangle + d\langle (0), (N')^\perp \rangle \\ &\quad + g \left[d\langle N^\perp \rangle - d\langle (N')^\perp \rangle \right] + (\Delta g)d\langle (N')^\perp \rangle \\ &= \left[(f(S) - f(S'))dC + d\langle \nu, \Delta N^\perp \rangle + gd\langle N^\perp + (N')^\perp, \Delta N^\perp \rangle \right] \\ &\quad + \left[(\Delta f)(S')dC + d\langle (\Delta \nu), (N')^\perp \rangle + (\Delta g)d\langle (N')^\perp \rangle \right] \\ &= \left[(f(S) - f(S'))dC + d\langle \nu', \Delta N^\perp \rangle \right] + \left[d(\Delta V)(S') \right] , \end{aligned}$$

where $\nu' = \nu + \int gd(N^\perp + (N')^\perp)$. By the assumptions on the coefficients, we know that $d(\Delta V)(S')_t = dI_t \geq 0$. So we find that against $1_{\{\Delta Y_t=0\}}$ we have

$$1_{\{\Delta Y_t=0\}} d\Delta V_t = 0 + 1_{\{\Delta Y_t=0\}} dI_t .$$

In the end,

$$1_{\{\Delta Y=0\}} \left(-dI - d\Delta K \right) = \frac{1}{2}dl ,$$

so

$$1_{\{\Delta Y=0\}} d\Delta K = -1_{\{\Delta Y=0\}} \underbrace{dI}_{\geq 0} - \frac{1}{2} \underbrace{dl}_{\geq 0} \leq 0 ,$$

and so we have proven that $1_{\{\Delta Y=0\}} d\Delta K \leq 0$. And when $\Delta Y > 0$, one has $Y > Y' \geq L' = L$ so $dK = 0 \leq dK'$, and therefore $1_{\{\Delta Y>0\}} d\Delta K \leq 0$, which completes the proof. \square

4 Existence.

We now turn to the existence theorem. We use the pertubation procedure like in Tevzadze [26] to construct a solution, under the assumption that the derivatives of f are controlled in the following way :

(**A_{der}**) There exists $\rho, \rho', \lambda > 0$, $r \in L^{\infty,2}$ and $h \in L^2_{BMO}$ such that

$$\begin{aligned} |f_y(t, y, z)| &\leq \rho r_t^2 \quad \text{and} \quad |f_z(t, y, z)| \leq \rho'(h_t + |z|) , \\ |f_{yy}(t, y, z)| &\leq \lambda r_t^2, \quad |f_{yz}(t, y, z)| \leq \lambda r_t \quad \text{and} \quad |f_{zz}(t, y, z)| \leq \lambda . \end{aligned}$$

4.1 Principle.

As said in the introduction, the strategy is to first apply the fixed point theorem. To perform this, one will use only the following assumption on f :

- (**A_{locLip}**) The function f is differentiable at $(0,0)$ (in (y,z) , for all (ω,s)), and there exist $\lambda > 0$ such that, writing $\beta_s = f_y(s,0,0)$ and $\gamma_s = f_z(s,0,0)$, one has
- for all $\omega, s, y_1, y_2, z_1, z_2$:

$$\begin{aligned} & |f(s, y_1, z_1) - f(s, y_2, z_2) - \beta_s(y_1 - y_2) - \gamma_s(z_1 - z_2)| \\ & \leq \lambda \left(r_s |y_1| + r_s |y_2| + |z_1| + |z_2| \right) (r_s |y_1 - y_2| + |z_1 - z_2|) , \end{aligned}$$

$$- \gamma \in L_{BMO}^2 \text{ and } \beta \in L^{\infty,1} \text{ (that is : } \int_0^T |\beta_s| dC_s \in L^\infty),$$

which follows naturally from the assumption on the second derivative of f only in (**A_{der}**). In all generality, it allows also for quadratic growth in y . So what one actually proves first is that when f satisfies this assumption (with possibly quadratic growth in y and z), and when the parameters are small enough (in a sense to specify), there exists a solution.

The perturbations procedure is then carried as follows for a reflected BSDE with obstacle $L \leq 0$. Split the initial data in n (equal) pieces : for $i \in \{1, \dots, n\}$, set $\xi^i = \frac{1}{n}\xi$ and $\alpha^i = \frac{1}{n}\alpha$ where $\alpha = f(\cdot, 0, 0)$, so that (ξ^i, α^i) is small enough. First, there is a solution $S^1 = (Y^1, N^1, K^1)$ to the reflected BSDE (2) with small parameters $(f - \alpha + \alpha^1, \nu, g, \xi^1, L)$.

Now, unless otherwise specified, we denote by \bar{x}^k the sum $\sum_{j=1}^k x^j$, for a general quantity x indexed by $\{1, \dots, n\}$. For $i = 2$ to n , having obtained a solution $\bar{S}^{i-1} = (\bar{Y}^{i-1}, \bar{N}^{i-1}, \bar{K}^{i-1})$ to the reflected BSDE (2) with parameters $(f - \alpha + \bar{\alpha}^{i-1}, \nu, g, \bar{\xi}^{i-1}, L)$, one incorporates one more (α^i, ξ^i) in the system. One first obtains the perturbation $S^i = (Y^i, N^i, K^i)$ solving the perturbation equation

$$\begin{cases} dY^i = -dV^i(Y^i, N^i) - dK^i + dN^i , \\ Y_T^i = \xi^i , \\ \bar{Y}^{i-1} + Y^i \geq \bar{L}^{i-1} + L^i , \\ d\bar{K}^{i-1} + dK^i \geq 0 \text{ and } (d\bar{K}^{i-1} + dK^i)(\bar{Y}^{i-1} + Y^i > \bar{L}^{i-1} + L^i) = 0 \end{cases} \quad (5)$$

with drift given by

$$\begin{aligned} dV^i(Y^i, N^i)_s &= [f(\bar{S}^{i-1} + S^i) - f(\bar{S}^{i-1}) + \alpha_s^i] dC_s + d\langle \nu + \int 2gd(\bar{N}^{i-1})^\perp, (N^i)^\perp \rangle_s + gd\langle (N^i)^\perp \rangle_s \\ &= [\bar{f}^{i-1}(S^i) + \alpha_s^i] dC_s + d\langle \bar{\nu}^{i-1}, (N^i)^\perp \rangle_s + gd\langle (N^i)^\perp \rangle_s , \end{aligned}$$

where $\bar{\nu}^{i-1} = \nu + \int 2gd(\bar{N}^{i-1})^\perp$, and \bar{f}^{i-1} is the translated/recentered function f around \bar{S}^{i-1} . It satisfies $\bar{f}^{i-1}(0) = 0$ so the residual-drift (constant part) in this equation is given by α^i . So the parameters $(\bar{f}^{i-1} + \alpha^i, \nu^{i-1}, g, \xi^i, L)$ here are small in the required sense. Finally, one sums $\bar{S}^i := \bar{S}^{i-1} + S^i$ to obtain a solution to the reflected BSDE (2) of parameters $(f - \alpha + \bar{\alpha}^i, \nu, g, \bar{\xi}^i, L)$. For $i = n$ this provides a solution to the reflected BSDE of interest.

This allows us to conclude to existence for those reflected BSDEs with negative obstacles. But then we can show that up to translation, this covers all the cases where the obstacle is upper-bounded.

Note that the above perturbation equation (5) is not a RBSDE in the variable $S^i = (Y^i, N^i, K^i)$ because K^i is not necessarily increasing. It could be viewed as a reflected BSDE in the variable $(Y^i, N^i, \overline{K}^i)$ but this point of view will not be used (see the remark after proposition 9 and remark 10 after its proof). Also, note that the solution S^1 to the initial, small RBSDE can be viewed as a perturbation : $\overline{S}^1 = 0 + S^1$, 0 being the solution to the RBSDE of parameters $(f - f(\cdot, 0, 0), \nu, g, 0, L)$. So it would be enough to study only the perturbation equations, but it seemed clearer to treat first the small reflected BSDEs and then deal with what changes for the perturbation equations.

4.2 Existence for small reflected BSDEs.

4.2.1 Underlying problem.

In order to use the fixed point theorem, we need to check that the underlying problem, that is to say the backward stochastic problem that one sees when the drift dV_t is a fixed process and doesn't depend on the solution, defines indeed a map from $\mathcal{S}^\infty \times BMO$ to itself. For reflected BSDEs, as was explained in El Karoui *et al.* [13], the solution is the Snell envelope of a certain process (more precisely, $Y + \int_0^\cdot dV_s$ is the Snell envelope of $L + \int_0^\cdot dV_s$). We want to check that it lives in the right space.

Proposition 6. *Let $V \in L^1_{BMO}$ (in the sense that $\sup_t \|E(\int_t^T |dV_s| | \mathcal{F}_t)\|_\infty < +\infty$), $\xi \in L^\infty$, and L be upper bounded. There exist a unique $(Y, N, K) \in \mathcal{S}^\infty \times BMO \times \mathcal{A}$ solution to the reflected BSDE :*

$$\begin{cases} dY = -dV - dK + dN , \\ Y_T = \xi , \\ Y \geq L \text{ and } 1_{Y > L} dK = 0 . \end{cases} \quad (6)$$

In particular, this applies when $dV_s = dV(y, n)_s = f(s, y_s, z_s \sigma_s) dC_s + d\langle \nu, n^\perp \rangle_s + g_s d\langle n^\perp \rangle_s$, for f satisfying the quadratic growth condition $(\mathbf{A}_{\mathbf{qg}})$, $\nu \in BMO$, $g \in L^\infty$ and $(y, n) \in \mathcal{S}^\infty \times BMO$.

Proof. We know from proposition 5.1 in El Karoui *et al.* [13], that Y_t is given by :

$$Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t^T} E \left(\int_t^\tau dV_s + L_\tau 1_{\tau < T} + \xi 1_{\tau = T} \middle| \mathcal{F}_t \right) , \quad (7)$$

where \mathcal{T}_t^T are the stopping times τ such that $t \leq \tau \leq T$, and that (K, N) is the Doob-Meyer decomposition of the supermartingale $Y + V$. Our goal is to check that (Y, N) is indeed in $\mathcal{S}^\infty \times BMO$.

For an upper bound on Y_t , we say that

$$\begin{aligned} E \left(\int_t^\tau dV_s + L_\tau 1_{\tau < T} + \xi 1_{\tau = T} \middle| \mathcal{F}_t \right) &\leq E \left(\int_t^T |dV_s| \middle| \mathcal{F}_t \right) + E \left(L_\tau^+ \middle| \mathcal{F}_t \right) + E \left(\xi^+ \middle| \mathcal{F}_t \right) \\ &\leq \|V\|_{L^1_{BMO}} + \|L^+\|_\infty + \|\xi^+\| \end{aligned}$$

for any stopping time τ , so $Y_t \leq \|V\|_{L^1_{BMO}} + \|L^+\|_\infty + \|\xi^+\|$. For a lower bound, we just say

that since Y solves (6), using the fact that K is increasing,

$$\begin{aligned} Y_t &= E\left(\xi + \int_t^T dV_s + (K_T - K_t) \middle| \mathcal{F}_t\right) \\ &\geq E\left(\xi + \int_t^T dV_s \middle| \mathcal{F}_t\right) \\ &\geq -\|\xi^-\|_\infty - \|V\|_{L^1_{BMO}} , \end{aligned}$$

so Y is indeed in \mathcal{S}^∞ .

One can then invoke remark 2 after proposition 1 to conclude that $N \in BMO$. \square

4.2.2 Existence for RBSDEs with small data.

First one proves that there is a solution when the data are small and when, essentially, the drift is purely quadratic in the solution.

Proposition 7. *Let $\lambda > 0$, $r \geq 0$ such that $r \in L^{\infty,2}$. Let f satisfy assumption $(\mathbf{A}_{\text{locLipz}})$, with parameters $(\beta = 0, \gamma = 0, \lambda, r)$ and be such $\alpha = f(\cdot, 0, 0) \in L^{\infty,1}$ (i.e. : $\int_0^T |\alpha_s| dC_s \in L^\infty$). Let $\nu = 0 \in BMO$ and g be bounded by λ . There exists $\epsilon_0 = \epsilon_0(\lambda, r) > 0$ such that if*

$$E = \|\xi\|_\infty + \|f(\cdot, 0, 0)\|_{L^{\infty,1}} + \|L^+\|_\infty \leq \epsilon_0 ,$$

then there exists a solution $S = (Y, N, K) \in \mathcal{S}^\infty \times BMO(P) \times \mathcal{A}$ to the reflected BSDE (2) with parameters (V, ξ, L) , where $dV(Y, N)_s = f(s, Y_s, Z_s \sigma_s) dC_s + g_s d\langle N^\perp \rangle_s$.

More precisely,

$$\epsilon_0(\lambda, r) = \frac{1}{2^{10} \lambda (\|r\|_{L^{\infty,2}}^2 + 2)} .$$

Also, for any $R \leq R_0(\lambda, r) = \frac{1}{2^5 \lambda (\|r\|^2 + 2)}$, if $E \leq \frac{R}{2^5}$, then this solution is known to satisfy

$$\|S\|^2 = \|Y\|_{\mathcal{S}^\infty}^2 + \|N\|_{BMO(P)}^2 \leq R^2 .$$

Proof. We study the map $Sol : \mathcal{S}^\infty \times BMO \rightarrow \mathcal{S}^\infty \times BMO$ which sends (y, n) on the solution (Y, N) to the reflected BSDE

$$\begin{cases} dY = -dV(y, n) - dK + dN , \\ Y_T = \xi , \\ Y \geq 0 \text{ and } 1_{\{Y > 0\}} dK = 0 , \end{cases} \quad (8)$$

where $dV(y, n)_s = f(s, y_s, z_s \sigma_s) dC_s + g_s d\langle n^\perp \rangle_s$. This map is well defined according to proposition 6, and $(Y, N) \in \mathcal{S}^\infty \times BMO$ is a solution of (2) if and only if it is a fixed point of Sol . It will be seen that Sol is not a contraction on the whole space, but it is on a small ball, and it stabilizes such a small ball if the data are small enough. Therefore there exists at least one fixed point in the space.

We study first the regularity of Sol . Take $s = (y, n)$ and $s' = (y', n')$ in $\mathcal{S}^\infty \times BMO$, write $S = Sol(s)$, $S' = Sol(s')$, and $\Delta x = x' - x$ for a generic quantity x . The semimartingale

decomposition of ΔY is $d\Delta Y = -d\Delta V - d\Delta K + d\Delta N$, and the terminal value is 0. Therefore, applying Itô's formula to $(\Delta Y)^2$ between $t \in \mathcal{T}_0^T$ and T , and taking the expectation conditional to \mathcal{F}_t one has, since $\int_0^\cdot \Delta Y d\Delta N$ is a martingale,

$$\begin{aligned} (\Delta Y_t)^2 + E\left(\int_t^T d\langle \Delta N \rangle_s | \mathcal{F}_t\right) &= 0^2 + 2E\left(\int_t^T \Delta Y_s d\Delta V_s | \mathcal{F}_t\right) \\ &\quad + 2E\left(\int_t^T \Delta Y_s d\Delta K_s | \mathcal{F}_t\right) - 0. \end{aligned} \quad (9)$$

Let us now look at the third term on the right-hand side. Using the fact that $Y dK = L dK$ and $Y' dK' = L dK'$ one has

$$\begin{aligned} \Delta Y d\Delta K &= (Y' - Y) dK' - (Y' - Y) dK \\ &= \underbrace{(L - Y)}_{\leq 0} \underbrace{dK'}_{\geq 0} - \underbrace{(Y' - L)}_{\geq 0} \underbrace{dK}_{\geq 0} \leq 0. \end{aligned}$$

Let us now deal with the second term :

$$E\left(\int_t^T \Delta Y_s d\Delta V_s | \mathcal{F}_t\right) \leq \|\Delta Y\|_\infty E\left(\int_t^T |d\Delta V_s| | \mathcal{F}_t\right).$$

The assumption on f gives

$$\begin{aligned} |d\Delta V_s| &\leq \lambda \left(r_s |y_s| + r_s |y'_s| + |z_s \sigma_s| + |z'_s \sigma_s| \right) (r_s |\Delta y_s| + |\Delta z_s \sigma_s|) dC_s \\ &\quad + |g_s| |d\langle \Delta n, n + n' \rangle_s|. \end{aligned}$$

Using the Cauchy-Schwartz and the Kunita-Watanabe inequalities, and the elementary inequality $(\sum_{i=1}^n a_i)^2 \leq n \sum a_i^2$, we have

$$\begin{aligned} E\left(\int_t^T |d\Delta V| | \mathcal{F}_t\right) &\leq 2^{\frac{3}{2}} \lambda \left(\|r\|^2 \|y'\|^2 + \|r\|^2 \|y\|^2 + \|z'\sigma\|^2 + \|z\sigma\|^2 \right)^{\frac{1}{2}} \left(\|r\|^2 \|\Delta y\|^2 + \|\Delta z\|^2 \right)^{\frac{1}{2}} \\ &\quad + \|g\| \|\Delta n^\perp\| \|(n')^\perp + n^\perp\|. \end{aligned}$$

By orthogonality, one has $\|z\|$ and $\|n^\perp\|$ smaller than $\|n\|$. By definition of the norm on $\mathcal{S}^\infty \times BMO$, $\|y\|^2 + \|n\|^2 = \|s\|^2$. This in particular implies that $\|n\| \leq \|s\|$. So, using $\|g\| \leq \lambda$, one finally has the estimate

$$E\left(\int_t^T |d\Delta V| | \mathcal{F}_t\right) \leq 2^{\frac{3}{2}} \lambda (\|r\|^2 + 1 + 1) (\|s'\| + \|s\|) \|\Delta s\|.$$

Equation (9) then yields, using $2ab \leq \frac{1}{4}a^2 + 4b^2$,

$$\begin{aligned} (\Delta Y_t)^2 + E\left(\langle \Delta N \rangle_T - \langle \Delta N \rangle_t | \mathcal{F}_t\right) &\leq \frac{1}{4} \|\Delta Y\|^2 \\ &\quad + 4 \times 2^3 \lambda^2 (\|r\|^2 + 2)^2 \times 2 (\|s'\|^2 + \|s\|^2) \|\Delta s\|^2, \end{aligned}$$

and by taking the sup, we finally find that, since $\|\Delta Y\| \leq \|\Delta S\|$,

$$\|\Delta S\|^2 \leq 2^8 \lambda^2 (\|r\|^2 + 2)^2 (\|s\|^2 + \|s'\|^2) \|\Delta s\|^2. \quad (10)$$

Let's now study the size of $S = Sol(s)$. Following the very same computations and arguments as for ΔS we have first

$$(Y_t)^2 + E\left(\int_t^T d\langle N \rangle_s | \mathcal{F}_t\right) \leq \|\xi\|^2 + 2E\left(\int_t^T Y_s dV_s | \mathcal{F}_t\right) + 2E\left(\int_t^T Y_s dK_s | \mathcal{F}_t\right). \quad (11)$$

Since $Y dK = L dK$ and, importantly, since K is increasing, one can write

$$\int_t^T Y dK = \int_t^T L dK \leq \|L^+\| (K_T - K_t) = \|L^+\| \left(Y_t - \xi - \int_t^T dV + (N_T - N_t) \right),$$

so that

$$E\left(\int_t^T Y dK | \mathcal{F}_t\right) \leq \|L^+\| \|Y_t\| + \|L^+\| \|\xi\| + \|L^+\| E\left(\int_t^T |dV| | \mathcal{F}_t\right) + 0.$$

Reinjecting this into (11) leads to

$$(Y_t)^2 + E\left(\langle N \rangle_{t,T} | \mathcal{F}_t\right) \leq \left(2\|\xi\|^2 + 10\|L^+\|^2\right) + \frac{1}{4}\|Y\|^2 + 9E\left(\int_t^T |dV| | \mathcal{F}_t\right)^2.$$

Now, by the assumption on f ,

$$|dV| \leq \left[f(s, 0, 0) + \lambda(r_s |y_s| + |z_s \sigma_s|)^2 \right] dC_s + |g| d\langle n \rangle$$

so

$$E\left(\int_t^T |dV| | \mathcal{F}_t\right) \leq \|f(\cdot, 0, 0)\|_{\infty,1} + 2\lambda(\|r\|^2 + 2)\|s\|^2$$

One obtains in the end

$$\|S\|^2 \leq 2^9 E^2 + 2^9 \lambda^2 (\|r\|^2 + 2)^2 \|s\|^4, \quad (12)$$

where $E = \|\xi\| + \|L^+\| + \|f(\cdot, 0, 0)\|_{\infty,1}$.

To have Sol be a contraction on a closed (and therefore complete) ball $\overline{B}(0, R)$ of $\mathcal{S}^\infty \times BMO$, we see from (10) and (12) that we would like R and E to be so small that $2^9 \lambda^2 (\|r\|^2 + 2)^2 R^2 \leq \frac{1}{2} (< 1)$ and $2^9 E^2 + 2^9 \lambda^2 (\|r\|^2 + 2)^2 R^4 \leq R^2$. This is the case as soon as

$$R \leq R_0(\lambda, r) = \frac{1}{2^5 \lambda (\|r\|^2 + 2)}$$

$$E \leq \frac{R}{2^5} \leq \frac{R_0(\lambda, r)}{2^5} =: \epsilon_0(\lambda, r).$$

□

We now remove the assumption that the linear terms in the drift are null.

Proposition 8. *Let $\lambda > 0$, $r \geq 0$ such that $r \in L^{\infty,2}$. Let f satisfy assumption $(\mathbf{A}_{\text{locLipz}})$, with parameters $(\beta, \gamma, \lambda, r)$ and be such that $\alpha = f(\cdot, 0, 0) \in L^{\infty,1}$ (i.e. : $\int_0^T |f(s, 0, 0)| dC_s \in L^\infty$). Let $\nu \in BMO$ and g be bounded by λ . There exists $\epsilon_0 = \epsilon_0(\beta, \lambda, r) > 0$ such that if*

$$E = \|\xi\|_\infty + \|f(\cdot, 0, 0)\|_{L^{\infty,1}} + \|L^+\|_\infty \leq \epsilon_0,$$

then there exists a solution $S = (Y, N, K) \in \mathcal{S}^\infty \times BMO(P) \times \mathcal{A}$ to the reflected BSDE (2) with parameters (f, ν, g, ξ, L) .

More precisely,

$$\epsilon_0(\beta, \lambda, r) = \frac{e^{-2\|\beta\|_{\infty,1}}}{2^{10}\lambda(\|r\|^2+2)}.$$

Also, for any $R \leq R_0(\hat{\lambda}, r) = \frac{1}{2^5\hat{\lambda}(\|r\|^2+2)}$, where $\hat{\lambda} = e^{\|\beta\|}\lambda$, if $E \leq \frac{e^{-\|\beta\|}R}{2^5}$, then this solution is known to satisfy

$$\|\hat{S}\|_Q^2 = \|\hat{Y}\|_{S^\infty}^2 + \|\hat{N}\|_{BMO(Q)}^2 \leq R^2,$$

where $\hat{Y}_t = e^{\int_0^t \beta_u dC_u} Y_t$ and \hat{N} is the martingale part of \hat{Y} under $Q : \frac{dQ}{dP} = \mathcal{E}(\int \gamma \sigma^{-1} dM + \nu)$.

Proof. Write $f(t, y, z) = \beta_t y + \gamma_t z + h(t, y, z)$, where $\beta_t = f_y(t, 0, 0)$ and $\gamma_t = f_z(t, 0, 0)$ (so that $h(t, 0, 0) = f(t, 0, 0) = \alpha_t$). Note that h satisfies $(\mathbf{A}_{\text{locLipz}})$ with parameters $(\beta = 0, \gamma = 0, \lambda, r)$.

The idea is that if (Y, N, K) is a solution to the reflected BSDE (2), one can eliminate the linear terms $(\beta_t Y_t + \gamma_t Z_t \sigma_t) dC_t + d\langle \nu, N^\perp \rangle_t$ in $dV(Y, N)_t$ by a pair of transforms and obtain a reflected BSDE with purely quadratic drift. Proposition 7 guarantees the existence of a solution to such a RBSDE, so undoing the transforms yields a solution to (2).

More precisely, define the measure Q by $\frac{dQ}{dP} = \mathcal{E}(L) = \mathcal{E}(\int \gamma \sigma^{-1} dM + \nu)$. Then $\widetilde{M} := M - \langle L, M \rangle = M - \int \gamma \sigma^* dC$ is a $BMO(Q)$ -martingale. Define also $B = \exp\left(\int_0^\cdot \beta_u dC_u\right)$, which is a bounded process. Define

$$\begin{aligned}\hat{h}(s, y, z) &= B_s h(s, B_s^{-1} y, B_s^{-1} z), \\ \hat{g}_s &= B_s^{-1} g_s, \\ \hat{\xi} &= B_T \xi, \\ \hat{L} &= BL.\end{aligned}$$

Note that \hat{h} satisfies $(\mathbf{A}_{\text{locLipz}})$ with parameters $(\beta = 0, \gamma = 0, \hat{\lambda}, r)$ where $\hat{\lambda} = \lambda e^{\|\beta\|}$. Proposition 7 ensures the existence of a solution $(\hat{Y}, \hat{N}, \hat{K}) \in \mathcal{S}^\infty \times BMO(Q) \times \mathcal{A}$ under Q to the reflected BSDE (2) with parameters $(\hat{h}, \nu = 0, \hat{g}, \hat{\xi}, \hat{L})$.

Indeed,

$$\begin{aligned}\|\hat{g}\|_\infty &\leq \exp(\|\beta\|)\|g\| \leq \exp(\|\beta\|)\lambda = \hat{\lambda} < +\infty, \\ \|\hat{h}(\cdot, 0, 0)\|_{\infty,1} &\leq \exp(\|\beta\|)\|f(\cdot, 0, 0)\| < +\infty, \\ \|\hat{\xi}\|_\infty &\leq \exp(\|\beta\|)\|\xi\| < +\infty, \\ \|\hat{L}^+\|_\infty &\leq \exp(\|\beta\|)\|L^+\| < +\infty,\end{aligned}$$

so if $E \leq \exp(-\|\beta\|)\epsilon_0(\hat{\lambda}, r) = \exp(-2\|\beta\|)\epsilon_0(\lambda, r)$, proposition 7 applies.

Define then $Y = B^{-1}\hat{Y}$, $\tilde{N} = \int_0^\cdot B^{-1}d\hat{N} = \int \tilde{Z}d\tilde{M} + \tilde{N}^\perp$ and $K = \int_0^\cdot B^{-1}d\hat{K}$. The Girsanov $(Q \rightarrow P)$ -transform of \tilde{N} ,

$$\begin{aligned}N &= \tilde{N} - \langle \tilde{L}, \tilde{N} \rangle \\ &= \int \tilde{Z}d\tilde{M} + \tilde{N}^\perp + \int \gamma \tilde{Z} \sigma dC + \langle \nu, \tilde{N}^\perp \rangle \\ &= \int \tilde{Z}dM + N^\perp,\end{aligned}$$

is a $BMO(P)$ -martingale. Y is a bounded semimartingale, since B^{-1} is bounded, and differentiating $Y = B^{-1}\hat{Y}$ shows that (Y, N, K) is solution to the reflected BSDE (2) with data (f, ν, g, ξ, L) (under P). \square

4.3 Perturbation of a reflected BSDE.

We now deal with existence for perturbation equations like (5). We assume we have a solution $S^1 = (Y^1, N^1, K^1)$ to a reflected BSDE, and want to construct a solution \bar{S}^2 to a reflected BSDE with slightly different parameters. The idea is to construct the difference $S^2 = (Y^2, N^2, K^2) = \bar{S}^2 - S^1$. The next proposition shows how this can be done despite the fact that K^2 is not an increasing process, so long as one does not change the obstacle.

Proposition 9. *Let f satisfy $(\mathbf{A}_{\text{der}})$ with parameters $(\rho, \rho', \lambda, r, h)$ and be such that $\alpha = f(\cdot, 0, 0) \in L^{\infty,1}$, let $g \in L^\infty$ be bounded by λ and $\nu \in BMO$. Let also $\xi^1 \in L^\infty$ and L^1 be upper bounded. Assume that there exists a solution $S^1 = (Y^1, N^1, K^1)$ to the RBSDE (2) with parameters (f, g, ν, ξ^1, L^1) . Now let $\xi^2 \in L^\infty$ and $\alpha^2 \in L^{\infty,1}$ (and $L^2 = 0$). If*

$$\delta E = \|\xi^2\|_\infty + \|\alpha^2\|_{L^{\infty,1}} \leq \epsilon_0(\rho, 2\lambda, r) = \frac{e^{-2\rho\|r\|^2}}{2^{10}(2\lambda)(\|r\|^2 + 2)},$$

then there exist $S^2 = (Y^2, N^2, K^2)$ where $Y^2 \in \mathcal{S}^\infty$, $N^2 \in BMO(P)$ and K^2 has finite variation, solving the perturbation equation

$$\begin{cases} dY^2 = -dV^2(Y^2, N^2) - dK^2 + dN^2, \\ Y_T^2 = \xi^2, \\ Y^1 + Y^2 \geq L^1 + L^2, \\ dK^1 + dK^2 \geq 0 \text{ and } 1_{\{Y^1 + Y^2 > L^1 + L^2\}}(dK^1 + dK^2) = 0 \end{cases} \quad (13)$$

with drift

$$\begin{aligned} dV_s^2 &= [f(s, Y_s^2 + Y_s^1, Z_s^2 \sigma_s + Z_s^1 \sigma_s) - f(s, Y_s^1, Z_s^1 \sigma_s) + \alpha_s^2] dC_s \\ &\quad + d\left\langle \nu + \int 2gd(N^1)^\perp, (N^2)^\perp \right\rangle_s + g_s d\left\langle (N^2)^\perp \right\rangle_s. \end{aligned}$$

So $\bar{S}^2 := S^1 + S^2$ is a solution to the RBSDE (2) with parameters $(f + \alpha^2, g, \nu, \xi^1 + \xi^2, L^1)$.

We further know that for any $R \leq R_0(\widehat{2\lambda}, r) = \frac{1}{2^{5/2}\widehat{\lambda}(\|r\|^2 + 2)}$, if $\delta E \leq \frac{e^{-\rho\|r\|^2}}{2^5} R$, then this solution satisfies

$$\|\widehat{S^2}\|_Q^2 = \|\widehat{Y^2}\|_{\mathcal{S}^\infty}^2 + \|\widehat{N^2}\|_{BMO(Q)}^2 \leq R^2.$$

Note that while $S^2 = (Y^2, N^2, K^2)$ is not the solution to a reflected BSDE, (Y^2, N^2, \bar{K}^2) is. However the drift there would be $dV_s^2 - dK_s^1$, whose residual action (when $(Y^2, N^2) = 0$) is $\alpha_s^2 dC_s - dK_s^1$, and this has no reason to be small. We can therefore not simply invoke proposition 8 for (Y^2, N^2, \bar{K}^2) and have to do some extra work.

Proof. The majority of computations that would need to be done here, related to the dynamics of Y^2 , are very similar to those in the proposition 8 about small RBSDEs, so we only do the part which is different.

Define $\bar{f}(s, y, z) = f(s, y + Y_s^1, z + Z_s^1 \sigma_s) - f(s, Y_s^1, Z_s^1 \sigma_s) + \alpha_s^2$. Note that since f satisfies $(\mathbf{A}_{\text{der}})$, \bar{f} satisfies $(\mathbf{A}_{\text{locLip}})$ with parameters $(\bar{\beta}, \bar{\gamma}, 2\lambda, r)$, where $\bar{\beta} = f_y(\cdot, Y^1, Z^1 \sigma)$ and $\bar{\gamma} = f_z(\cdot, Y^1, Z^1 \sigma)$. We have $\|\bar{\beta}\|_{\infty,1} \leq \rho\|r\|^2 < +\infty$ and $\bar{\gamma} \in L_{BMO}^2$.

Following the same approach as for RBSDEs, we first look at the underlying perturbation problem of finding (Y^2, N^2, K^2) solving the perturbation equation (13) when

$$dV_s^2 = [f(s, y_s^2 + Y_s^1, z_s^2 \sigma_s + Z_s^1 \sigma_s) - f(s, Y_s^1, Z_s^1 \sigma_s) + \alpha_s^2] dC_s \\ + d\left\langle \nu + \int 2gd(N^1)^\perp, (n^2)^\perp \right\rangle_s + g_s d\left\langle (n^2)^\perp \right\rangle_s.$$

Because $\bar{S}^2 = S^1 + S^2$ is then solution to the reflected BSDE (8) (or rather (6)) with drift $d\bar{V}_s^2 = dV^1(Y^1, N^1)_S + dV^2(y^2, o^2)_s = [f(s, y_s^2 + Y_s^1, z_s^2 \sigma_s + Z_s^1 \sigma_s) + \alpha_s^2] dC_s + d\langle \nu, N^1 + n^2 \rangle_s + g_s d\langle N^1 + n^2 \rangle_s$, proposition 6 guarantees the existence and uniqueness of \bar{S}^2 and therefore of S^2 . This allows to define a map Sol' from $\mathcal{S}^\infty \times BMO$ to itself.

Now, to find a solution S^2 to the perturbation equation (13), we proceed like in propositions 7 and 8, the difference being in dealing with dK^2 which is not monotonous anymore here. Up to doing the usual transformations (proposition 8), let us assume that the drift is purely quadratic as in proposition 7. Then, Itô's formula first leads to the estimates

$$|\Delta Y_t^2|^2 + E\left(\langle \Delta N^2 \rangle_{t,T} \middle| \mathcal{F}_t\right) \leq 2E\left(\int_t^T \Delta Y_s^2 d\Delta V_s^2 \middle| \mathcal{F}_t\right) + 2E\left(\int_t^T \Delta Y_s^2 d\Delta K_s^2 \middle| \mathcal{F}_t\right), \\ |Y_t^2|^2 + E\left(\langle N^2 \rangle_{t,T} \middle| \mathcal{F}_t\right) \leq \|\xi^2\|^2 + 2E\left(\int_t^T Y_s^2 dV_s^2 \middle| \mathcal{F}_t\right) + 2E\left(\int_t^T Y_s^2 dK_s^2 \middle| \mathcal{F}_t\right).$$

For the term in $\Delta Y^2 d\Delta K^2$ one has (even if $L^2 \neq 0$)

$$\Delta Y^2 d\Delta K^2 = (Y^{2'} - Y^2) dK^{2'} - (Y^{2'} - Y^2) dK^2 \\ = (\bar{Y}^{2'} - \bar{Y}^2)(d\bar{K}^{2'} - dK^1) - (\bar{Y}^{2'} - \bar{Y}^2)(d\bar{K}^2 - dK^1) \\ = (\bar{Y}^{2'} - \bar{Y}^2) d\bar{K}^{2'} - (\bar{Y}^{2'} - \bar{Y}^2) d\bar{K}^2 \\ = \underbrace{(\bar{L}^2 - \bar{Y}^2)}_{\leq 0} \underbrace{d\bar{K}^{2'}}_{\geq 0} - \underbrace{(\bar{Y}^{2'} - \bar{L}^2)}_{\geq 0} \underbrace{d\bar{K}^2}_{\geq 0} \leq 0.$$

For the term in $Y^2 dK^2$ one has however, since $L^2 = 0$,

$$Y^2 dK^2 = (Y^2 - L^2) dK^2 + L^2 dK^2 \\ = ((\bar{Y}^2 - Y^1) - (\bar{L}^2 - L^1)) dK^2 + L^2 dK^2 \\ = ((\bar{Y}^2 - \bar{L}^2) - (Y^1 - L^1))(d\bar{K}^2 - dK^1) + L^2 dK^2 \\ = \underbrace{(\bar{Y}^2 - \bar{L}^2) d\bar{K}^2}_{=0} - \underbrace{(\bar{Y}^2 - \bar{L}^2) dK^1}_{\geq 0} - \underbrace{(Y^1 - L^1) d\bar{K}^2}_{\geq 0} + \underbrace{(Y^1 - L^1) dK^1}_{=0} + L^2 dK^2 \\ \leq L^2 dK^2 = 0.$$

Having observed this, the rest is like the analysis of the map Sol and the ϵ_0 is the same. So in the end, provided that $\delta E = \|\xi^2\|_\infty + \|\alpha^2\|_{L^\infty, 1} \leq \epsilon_0(\rho, 2\lambda, r) = \frac{e^{-2\rho\|r^2\|}}{2^{10}(2\lambda)(\|r\|^2+2)}$, there exists a solution (Y^2, N^2, K^2) to the perturbation equation (13). \square

Remark 10. Note for later use that uniqueness holds for the perturbation equations. First, under $(\mathbf{A}_{\text{der}})$, $(\mathbf{A}_{\text{Lip}})$ holds and so does uniqueness for reflected BSDEs. Then, one can argue that if Y^2 and Y'^2 are two solutions to (13), then $\bar{Y}^2 = Y^1 + Y^2$ and $\bar{Y}'^2 = Y^1 + Y'^2$ are two solutions to the same reflected BSDE, so $\bar{Y}^2 = \bar{Y}'^2$ and therefore $Y^2 = Y'^2$. Or we can also argue that if (Y^2, N^2, K^2) is a solution to (13), then (Y^2, N^2, \bar{K}^2) solve a reflected BSDE (2) for which uniqueness holds.

4.4 Existence theorem.

We can now prove the existence theorem.

Theorem 11. *Let f satisfy $(\mathbf{A}_{\text{der}})$ with parameters $(\rho, \rho', \lambda, r, h)$ and be such that $f(\cdot, 0, 0) \in L^{\infty, 1}$. Let $\nu \in BMO$, $g \in L^\infty$ be bounded by λ , $\xi \in L^\infty$, and L be upper bounded. There exists a solution $(Y, N, K) \in \mathcal{S}^\infty \times BMO \times \mathcal{A}$ to the RBSDE (2) with parameters (f, g, ν, ξ, L) .*

Proof. The proof is done in two steps. First, we show that one can indeed reduce the problem to the case $L \leq 0$, by translation. Existence for the RBSDE with $L \leq 0$ is then proved by repeatedly perturbing a solution to a similar RBSDE with smaller data.

Step 1. If (Y, N, K) is a solution to the RBSDE, and U is an upper bound for L , set $\vec{Y} = Y - U$. We see that

$$\begin{cases} d\vec{Y} = dY - dU = -dV - dK + dN - 0 = -d\vec{V} - dK + dN, \\ \vec{Y}_T = \xi - U =: \vec{\xi}, \\ \vec{Y} = Y - U \geq L - U =: \vec{L}, \\ 1_{\{\vec{Y} > L - U\}} dK = 1_{\{Y > L\}} dK = 0. \end{cases}$$

Here

$$\begin{aligned} d\vec{V} &= dV(Y, N) \\ &= dV(\vec{Y} + U, N) \\ &= f(s, \vec{Y} + U, Z\sigma) dC_s + d\langle \nu, N^\perp \rangle + g d\langle N^\perp \rangle \\ &= \vec{f}(s, \vec{Y}, Z\sigma) dC_s + d\langle \nu, N^\perp \rangle + g d\langle N^\perp \rangle. \end{aligned}$$

It is clear that \vec{f} still satisfies $(\mathbf{A}_{\text{der}})$ with parameters $(\rho, \rho', \lambda, r, h)$. And from the assumption on f_y one has

$$|\vec{f}(s, 0, 0)| = |f(s, U, 0)| \leq |f(s, 0, 0)| + \rho r^2 U,$$

so $\vec{\alpha} = \vec{f}(\cdot, 0, 0) \in L^{\infty, 1}$.

In the end, $(\vec{Y}, N, K) \in \mathcal{S}^\infty \times BMO \times \mathcal{A}$ is a solution to the reflected BSDE of parameters $(\vec{f}, \nu, g, \vec{\xi}, \vec{L})$ satisfying the same assumptions, but with $\vec{L} \leq 0$.

Step 2. We now focus on the case $L \leq 0$. Consider ϵ_0 given by proposition 9 (or proposition 8, it's the same). Choose an integer n big enough so that, setting $\alpha^i = \frac{1}{n}\alpha$ (where $\alpha = f(\cdot, 0, 0)$) and $\xi^i = \frac{1}{n}\xi$, one has $E^i = \|\xi^i\| + \|\alpha^i\| = \frac{1}{n}E \leq \epsilon_0$.

First, by proposition 8, there exists a solution (Y^1, N^1, K^1) to the RBSDE (2) with parameters $(f - \alpha + \alpha^1, \nu, g, \xi^1, L)$.

Next, for $i = 2$ to n , having obtained a solution $(\bar{Y}^{i-1}, \bar{N}^{i-1}, \bar{K}^{i-1})$ to the RBSDE (2) with parameters $(f - \alpha + \bar{\alpha}^{i-1}, \nu, g, \bar{\xi}^{i-1}, L)$, proposition 9 provides a solution (Y^i, N^i, K^i) to the perturbation equation (5) and therefore a solution $(\bar{Y}^i, \bar{N}^i, \bar{K}^i)$ to the RBSDE (2) with parameters $(f - \alpha + \bar{\alpha}^i, \nu, g, \bar{\xi}^i, L)$. For $i = n$, since $\bar{\xi}^n = \xi$ and $\bar{\alpha}^n = \alpha$, $(\bar{Y}^n, \bar{Z}^n, \bar{K}^n)$ is a solution to the RBSDE of interest, which ends the proof. \square

4.5 Stability in $\mathcal{S}^\infty \times BMO$.

Given that uniqueness holds, the a posteriori bounds that come with the construction of a perturbation $S' - S$ to a solution S in proposition 9 readily give the continuity of the map $(\xi, \alpha) \mapsto (Y, N)$, from $L^\infty \times L^{\infty,1}$ to $\mathcal{S}^\infty \times BMO$.

We now derive an estimate which shows that it is locally Lipchitz, by a sort of bootstrap argument on the above stability result. We use the notation $\delta S' = S' - S$, $\delta S'' = S'' - S$ for perturbations around a fixed solution S and $\Delta S = S'' - S' = \delta S'' - \delta S'$.

Proposition 12. *Suppose that f satisfies $(\mathbf{A}_{\text{der}})$ with parameters $(\rho, \rho', \lambda, r, h)$, that $\alpha = f(\cdot, 0, 0) \in L^{\infty,1}$, that $\nu \in BMO$, that g is bounded by λ and that L is upper bounded. We consider $\xi \in L^\infty$ and the solution (Y, N, K) to the reflected BSDE of parameters (f, ν, g, ξ, L) .*

Now, for any $(\xi', \delta\alpha')$ and $(\xi'', \delta\alpha'') \in L^\infty \times L^{\infty,1}$, let $S' = (Y', N', K')$ and $S'' = (Y'', N'', K'')$ be the solutions to the reflected BSDEs of parameters $(f + \delta\alpha', \nu, g, \xi', L)$ and $(f + \delta\alpha'', \nu, g, \xi'', L)$. If

$$\delta E', \delta E'' = \|\delta\xi'\| + \|\delta\alpha'\| \leq \frac{1}{\sqrt{2}} \epsilon_0(\bar{\beta}, 2\lambda, r) = \frac{1}{\sqrt{2}} \frac{e^{-2\|\bar{\beta}\|}}{2^{10}(2\lambda)(\|r\|^2 + 2)},$$

where $\bar{\beta} = f_y(\cdot, Y, Z\sigma)$, then we have

$$\begin{aligned} \|Y'' - Y'\|_{\mathcal{S}^\infty} &\leq 2^5 e^{2\|\bar{\beta}\|} (\|\xi'' - \xi'\|_\infty + \|\alpha'' - \alpha'\|_{\infty,1}) \quad \text{and} \\ \|N'' - N'\|_{BMO(P)} &\leq 2^5 e^{2\|\bar{\beta}\|} C(Y, N) (\|\xi'' - \xi'\|_\infty + \|\alpha'' - \alpha'\|_{\infty,1}), \end{aligned}$$

where $C(Y, N)$ is a constant depending on (Y, N) .

Proof. We know that $\bar{f}(s, \delta y, \delta z) = f(s, Y_s + \delta y, Z_s \sigma_s + \delta z) - f(s, Y_s, Z_s \sigma_s)$ satisfies $(\mathbf{A}_{\text{locLip}})$ with parameters $(\bar{\beta}, \bar{\gamma}, 2\lambda, r)$, where $\bar{\beta} = f_y(\cdot, Y, Z\sigma)$ and $\bar{\gamma} = f_z(\cdot, Y, Z\sigma)$ satisfy $\|\bar{\beta}\|_{\infty,1} \leq \rho\|r\|^2$ and $\bar{\gamma} \in L^2_{BMO}$; and $\bar{\nu} = \nu + \int 2gdN \in BMO$. We linearize \bar{f} like in proposition 8 : $\bar{f}(s, \delta y, \delta z) = \bar{\beta}_s \delta y + \bar{\gamma}_s \delta z + \bar{h}(s, \delta y, \delta z)$.

Since $\Delta Y = Y'' - Y'$ has the dynamics

$$\begin{aligned} d\Delta Y_s &= -[\Delta\alpha_s + \bar{\beta}_s \Delta Y_s + \bar{\gamma}_s \Delta Z_s \sigma_s + \{\bar{h}(s, \delta Y''_s, \delta Z''_s \sigma_s) - \bar{h}(s, \delta Y'_s, \delta Z'_s \sigma_s)\}] dC_s \\ &\quad - d\langle \bar{\nu}, (\Delta N)^\perp \rangle_s - g_s d\langle (\delta N'')^\perp + (\delta N')^\perp, (\Delta N)^\perp \rangle_s - d\Delta K_s + d\Delta N_s, \end{aligned}$$

doing the usual transformations, with $\frac{dQ}{dP} = \mathcal{E}(\int \bar{\gamma} \sigma^{-1} dM + \bar{\nu})$ and $\bar{B} = e^{\int_0^\cdot \bar{\beta}_u dC_u}$ and the standard computations give, like in (10),

$$\|\widehat{\Delta S}\|_Q^2 \leq 2^9 \widehat{\Delta E} + 2^9 \widehat{2\lambda}^2 (\|r\|^2 + 2)^2 (\|\widehat{\delta S''}\|_Q^2 + \|\widehat{\delta S'}\|_Q^2) \cdot \|\widehat{\Delta S}\|_Q^2$$

But $\delta S''$ and $\delta S'$ are the unique solutions to the perturbation equations with parameters $(\bar{f} + \delta\alpha'', \bar{\nu}, g, \delta\xi'', L)$ and $(\bar{f} + \delta\alpha', \bar{\nu}, g, \delta\xi', L)$, and by the way they were constructed in proposition 9 (recall that $\delta E', \delta E'' \leq \frac{e^{-\|\bar{\beta}\|}}{2^5} \frac{R_0(2\lambda, r)}{\sqrt{2}}$) we know that they satisfy

$$\begin{aligned} \|\widehat{\delta S''}\|_Q^2, \|\widehat{\delta S'}\|_Q^2 &\leq \frac{R_0(2\lambda, r)^2}{2}, \quad \text{so} \\ \|\widehat{\delta S''}\|_Q^2 + \|\widehat{\delta S'}\|_Q^2 &\leq R_0(2\lambda, r)^2 = \frac{1}{2^{10} \widehat{2\lambda}^2 (\|r\|^2 + 2)^2}. \end{aligned}$$

Reinjecting this in the previous estimate we have $\|\widehat{\Delta S}\|_Q^2 \leq 2^9 \widehat{\Delta E} + \frac{1}{2} \|\widehat{\Delta S}\|_Q^2$ and therefore

$$\|\widehat{\Delta S}\|_Q^2 = \|\widehat{\Delta Y}\|_{\mathcal{S}^\infty}^2 + \|\widehat{\Delta N}\|_{BMO(Q)}^2 \leq 2^{10} \widehat{\Delta E}.$$

Then this implies that $\|\widehat{\Delta Y}\| \leq 2^5 \widehat{\Delta E}$ and so $\|\Delta Y\| \leq 2^5 e^{2\|\bar{\beta}\|} \Delta E$. For the same reason, $\|\widehat{\Delta N}\|_{BMO(Q)} \leq 2^5 e^{2\|\bar{\beta}\|} \Delta E$. By theorem 3.6 in Kazamaki, $\|\Delta N\|_{BMO(P)} \leq C(Q) \|\widehat{\Delta N}\|_{BMO(Q)}$ where the constant depends only on Q , or equivalently on the martingale $\int \bar{\gamma} \sigma^{-1} dM + \bar{\nu}$, and *in fine* on (Y, N) . \square

Note that the interesting part of the above result is only the martingale estimate. Indeed, the estimate for $Y'' - Y'$ in \mathcal{S}^∞ actually holds for any size of parameters (as can be seen easily by linearizing the drift, doing a change of measure to get rid of all the terms in N and solving for Y). As mentionned in the introduction, we know that $(\xi, \alpha) \mapsto N$ is global Lipschitz in \mathcal{H}^p , and $\frac{1}{2}$ -Hölder in BMO . The above estimate shows it is in fact locally Lipschitz in BMO .

5 Existence under more general assumptions.

In section 4, the existence of a solution was proved under the usual assumption that f is a Lipschitz function of y , and therefore at most linear in y , while being at most quadratic in z . In this section, we deal with more general assumptions on f .

To some extent, we would like to replace the constants ρ, ρ', λ in $(\mathbf{A}_{\text{der}})$ by arbitrary growth functions (while, of course, still assuming that f ends up with a growth in y compatible with existence of solutions). Looking back at proposition 9, we see that when ρ is a growth function, the maximal size ϵ allowed for a perturbation $(\delta\xi, \delta\alpha)$ of the parameters would depend on the size $\|Y^1\|$ of the solution. It is therefore not guaranteed that one can choose ϵ_0 uniformly for the perturbation procedure in the proof of theorem 11, or to put things differently, that a sequence of perturbations could terminate in finitely many steps. This however can be guaranteed if one can obtain an *a priori* bound for the solutions Y 's of reflected BSDEs with drift (f, ν, g) .

Superlinear growth in y .

We show in the following proposition how this would work when f is allowed to have slightly superlinear growth in y .

Proposition 13. *Consider a set of parameters (f, ν, g, ξ, L) satisfying the assumptions of theorem 11, but with ρ, ρ', λ in $(\mathbf{A}_{\text{der}})$ being growth function instead of constants. Further assume that $|f(t, y, 0)| \leq |f(t, 0, 0)| + \varphi(y)$ for a growth function φ such that $\int_1^{+\infty} \frac{1}{\varphi(y)} dy = +\infty$. Then there exists a solution (Y, N, K) to the reflected BSDE (2) with parameters (f, ν, g, ξ, L) .*

Proof. We will apply the perturbation procedure as was done previously when ρ, ρ', λ were constants.

First, by the estimate in theorem 1 in Kobylanski *et al.* [18], we know that there exists a function F increasing (a growth function) such that for any set of parameters (f, ν, g, ξ, L) satisfying the assumptions and for any solution (Y, N, K) we have $\|Y\| \leq F(\|\xi\|, \|\alpha\|)$. Now, for a fixed set of parameters, we define $\rho_m = \rho(F(\|\xi\|, \|\alpha\|))$. We fix n big enough that

$$\frac{E}{n} \leq \frac{e^{-2\rho_m \|r\|^2}}{2^{10}(2\lambda(1))(\|r\|^2 + 2)} = \epsilon_0(\rho_m, 2\lambda(1), r) .$$

We will construct n solutions S^i of reflected BSDEs or perturbation equations such that for each equation, the size of the data is $\frac{E}{n}$ and the size of the solution is such that $\|\widehat{Y}^i\| \leq 1$. Note that the $\widehat{}$ here indicates the multiplication by $B^i = \exp(\int_0^\cdot f_y(\bar{S}_u^{i-1}) dC_u)$.

We know that we can do a transation to be reduced to the case $L \leq 0$ so we assume from now on that $L \leq 0$. Define $\xi^i = \frac{1}{n}\xi$ and $\alpha^i = \frac{1}{n}\alpha$.

For $i = 1$, we first build a solution $S^1 = (Y^1, N^1, K^1)$ to the reflected BSDE (2) with parameters $(f - \alpha + \alpha^1, \nu, g, \xi^1, L)$. Proposition 8 as it is stated doesn't strictly apply, but we can adapt the proof. We define the integrating factor $B = e^{\int \beta dC}$ with $\beta = \bar{\beta}^0 = f_y(\cdot, 0, 0) \in L^{\infty,1}$ and the new measure Q by $\frac{dQ}{dP} = \mathcal{E}(\int \gamma \sigma^{-1} dM + \nu)$ where $\gamma = \bar{\gamma}^0 = f_z(\cdot, 0, 0) \in L^2_{BMO}$. Then, like in proposition 7, we look for a solution $(\widehat{Y}^1, \widehat{N}^1, \widehat{K}^1)$ to the reflected BSDE with no linear term by the fixed point theorem. We look for a solution in a ball of radius R and now further demand that $R \leq 1$, so that the conditions to be met are that

$$R \leq R_0(\widehat{2\lambda(1)}, r) = \frac{1}{2^{10}\widehat{2\lambda(1)}(\|r\|^2 + 2)} \text{ and } \widehat{E}_1 = \frac{\widehat{E}}{n} \leq \frac{R_0(\widehat{2\lambda(1)}, r)}{2^5} = \epsilon_0(\widehat{2\lambda(1)}, r),$$

where $\widehat{2\lambda(1)} = e^{\|\bar{\beta}^0\|}(2\lambda(1))$. Now, since we have chosen n such that $\frac{E}{n} \leq \frac{e^{-2\rho_m\|r\|^2}}{2^{10}(2\lambda(1))(\|r\|^2 + 2)}$,

$$\frac{\widehat{E}}{n} \leq e^{\|\bar{\beta}^0\|} \frac{E}{n} \leq \frac{e^{\|\bar{\beta}^0\|} e^{-2\rho_m\|r\|^2}}{2^{10}(2\lambda(1))(\|r\|^2 + 2)} = \frac{e^{2\|\bar{\beta}^0\|} e^{-2\rho_m\|r\|^2}}{2^{10}\widehat{2\lambda(1)}(\|r\|^2 + 2)} \leq \frac{1}{2^{10}\widehat{2\lambda(1)}(\|r\|^2 + 2)} = \epsilon_0(\widehat{2\lambda(1)}, r)$$

because $\|\bar{\beta}^0\| \leq \rho(0)\|r\|^2$ and $\rho(0) \leq \rho_m$. So we indeed get a solution $(\widehat{Y}^1, \widehat{N}^1, \widehat{K}^1)$ and doing the reverse transforms gives a solution $S^1 = (Y^1, N^1, K^1)$ to the reflected BSDE with the linear terms.

For $i = 2 \dots n$, we have a solution \bar{S}^{i-1} to the reflected BSDE (2) with parameters $(f - \alpha + \bar{\alpha}^{i-1}, \nu, g, \bar{\xi}^{i-1}, L)$ and want to construct the appropriate perturbation S^i . We simply do the same computations as in proposition 9, using the integrating factor $\bar{B}^{i-1} = e^{\int \bar{\beta}^{i-1} dC}$ where $\bar{\beta}^{i-1} = f_y(\cdot, \bar{Y}^{i-1}, \bar{Z}^{i-1} \sigma)$, and the change of measure $\frac{dQ}{dP} = \mathcal{E}(\int \bar{\gamma}^{i-1} \sigma^{-1} dC + \bar{\nu}^{i-1})$ where $\bar{\gamma}^{i-1} = f_z(\cdot, \bar{Y}^{i-1}, \bar{Z}^{i-1} \sigma)$ and $\bar{\nu}^{i-1} = \nu + \int 2gd(\bar{N}^{i-1})^\perp$.

Because we know by the a priori estimate on solutions of the reflected BSDE that $\|\bar{Y}^{i-1}\| \leq F(\|\frac{(i-1)}{n}\xi\|, \|\frac{(i-1)}{n}\alpha\|) \leq F(\|\xi\|, \|\alpha\|)$, we know that $\|\bar{\beta}^{i-1}\| \leq \rho(\|\bar{Y}^{i-1}\|)\|r\|^2 \leq \rho_m\|r\|^2$. Therefore, just as above, we find that the size \widehat{E}_i of the data is indeed small enough, and so we can produce a perturbation S^i . \square

As can be seen from the proof, the key to the generalization is to have an a priori estimate $\|Y\|_{\mathcal{S}^\infty} \leq F(\|\xi\|_\infty, \|\alpha\|_{\infty,1})$ for some growth function F .

Monotonicity condition with arbitrary growth in y .

We can also generalize the result of theorem 11 to the case where f is so-called monotonous (or 1-sided Lipschitz) in y , with arbitrary growth.

Proposition 14. *Consider a set of parameters (f, ν, g, ξ, L) satisfying the assumptions of theorem 11, but with ρ, ρ', λ in $(\mathbf{A}_{\text{der}})$ being growth function instead of constants.*

Further assume that $|f(t, y, 0)| \leq |f(t, 0, 0)| + \varphi(y)$ for a growth function φ and that there exists a constant μ such that for all y, y', z, s, ω ,

$$(y' - y)(f(s, y', z) - f(s, y, z)) \leq \mu r_s^2 |y' - y|^2$$

Then there exists a solution (Y, N, K) to the reflected BSDE (2) with parameters (f, ν, g, ξ, L) .

As seen above, it is enough to have an a priori estimate for Y . One can invoke the one obtained in the proof of theorem in 3.1 in [28]. Alternatively, having argued that it is enough to study the case where the obstacle is negative, one can linearize the driver in the N variable,

and do a measure change. Then, using Itô with $|\cdot|^2$ to take advantage of the monotonicity condition one could conclude that

$$\|Y\|_\infty^2 \leq 2e^{4\mu\|r\|^2} (\|\xi\|_\infty^2 + 2\|\alpha\|_{\infty,1}^2) = F(\|\xi\|, \|\alpha\|)^2$$

6 Conclusion.

We studied quadratic reflected BSDEs in a general continuous-martingale setting, and obtained the basic theorems (comparison and uniqueness, existence, stability) in that framework.

We showed how the comparison theorem was easily obtained via the standard linearization technique for BSDEs, and under only a local Lipschitz assumption on f . We also gave a new proof to the special comparison theorem for reflected BSDEs, which does not rely on comparison for BSDEs (through the penalization approach) and therefore does not require smoothness of f . This is understandable as this theorem is rather linked to the minimality condition of nonlinear Snell envelopes than on a regularity property.

We then adapted to reflected BSDE the technique introduced by Tevzadze in [26]. The first part, where one uses the fixed point theorem to construct a solution for RBSDE with small data, essentially controls the superlinearities by imposing *a priori* some bounds on the sought solution. It would work for any kind of superlinear growth of f in the variable y . It would even hold for superlinear-in- z coefficients f if $f_z(\cdot, 0, 0) \in L_{BMO}^2$ in fact. The perturbation part required a special attention since perturbations to a reflected BSDE don't satisfy a reflected BSDE (this is linked to the underlying problem for reflected BSDEs, a Snell envelope problem, being non-linear ; the technique would work more easily for non-linear equations for which the underlying problem is linear). The problems with the difference of increasing processes not being increasing can be avoided if one does not perturb the obstacle. We therefore applied first a transformation to the RBSDE to be led to study only the case where $L \leq 0$. We expect that this approach would also work for doubly reflected BSDEs if they can be reduced to studying the case where the lower obstacle L is negative and the upper obstacle U is positive.

We then obtained a local Lipschitz regularity estimate for the martingale part N of the solution, in the space BMO , which improves on the previously known regularity. The key in obtaining this estimate is essentially a bootstrapping of an existing stability result for $(Y, N) \in \mathcal{S}^\infty \times BMO$. While we used in proposition 12 the *a posteriori* estimate coming from using the fixed point theorem, the proof could be adapted to make use instead of an *a priori* estimate such as the one given in Kazi-Tani, Possamai and Zhou [16], or even just the qualitative continuity in $\mathcal{S}^\infty \times BMO$.

When f is Lipschitz in y , the series of perturbations can be carried uniformly without using an *a priori* bound on Y , since it was assumed that f_y was bounded. We showed that the perturbation procedure could still be used to construct solutions when f is not Lipschitz, so long as one can get an *a priori* estimate for $\|Y\|_\infty$. This allowed us to recover the cases where f can be slightly superlinear in y or monotonous with arbitrary growth.

Finally we note that the essential ideas here are not specific to backward stochastic problems, and could apply to a certain number of nonlinear equations.

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